ELSEVIER Journal of Pure and Applied Algebra 111 (1996) 213–228

JOURNAL OF
PURE AND
APPLIED ALGEBRA

Cohomological nonvanishing for modules over discrete groups [1]

Daniel Juan-Pineda [2]

Mathematics Department, University of Wisconsin-Madison, Madison, WI 53706, USA

Communicated by K. Gruenberg; received 18 January 1994; revised 1 February 1995

Abstract

Let Γ be a discrete group of finite virtual cohomological dimension, and let V be a f.g. Z-torsion free Γ module. In this situation, the Farrell cohomology of Γ with coefficients in V is defined and we prove in this paper cohomological non-vanishing results for these groups similar to those existing for finite groups and Tate cohomology, i.e. that we either have that these groups vanish for all dimensions or there are infinitely many nontrivial. The proof is based on a geometric approach to these groups, the study of Euler characteristics, minimal resolutions, and the notion of exponent.

1991 Math. Subj. Class.: 55

0. Introduction

In this paper we extend nonvanishing cohomological results for modules over finite groups to modules over discrete groups Γ, in a family $\mathscr{X}$ suitably defined, which are finitely generated over $\mathbb{Z}$. We introduce a numerical invariant $\mathcal{T}_\Gamma(V)$ which determines cohomological nonvanishing under certain hypotheses (see Proposition 2.7). We prove:

Theorem 5.23. *Let Γ be an $\mathscr{X}$-group with $\Gamma' \lhd \Gamma$ a torsion-free normal subgroup of finite index. Let $G := \Gamma/\Gamma'$. Let V be a finitely $\mathbb{Z}$-generated $\mathbb{Z}$-torsion-free*

[1] The results in this paper form a part of a doctoral dissertation to be presented at the University of Wisconsin-Madison.

[2] Supported by a scholarship from the Universidad Nacional Autónoma de México. Present address: IMATE-Morelia, Nicolás Romero 150, col. Centro. Morelia, Mich. Mexico 58000.

SSDI 0022-4049(95)00096-8

Γ-module. Assume that Γ is not torsion-free, and that $|G| = p^n$ for some prime p. Then either

$$dim_{\mathbb{F}_p} \widehat{H}^i(\Gamma, V) \otimes \mathbb{F}_p + (-1)^{i+1} \frac{\Upsilon_\Gamma(V)}{|\Gamma : \Gamma'|} > 0 \quad \forall i \in \mathbb{Z},$$

or

$$\widehat{H}^*(\Gamma, V) = 0 \quad \forall * \in \mathbb{Z}.$$

(Here $\widehat{H}^*$ denotes the Farrell cohomology.)

As an application of this result we have the following which describes a global nonvanishing for Farrell cohomology:

Theorem 5.24. *Under the hypotheses of the above theorem, precisely one of the following must hold:*

(1) *$\Upsilon_\Gamma(V) = 0$ and V is Γ- cohomologically trivial, or*
(2) *$\Upsilon_\Gamma(V) = 0$ and $\widehat{H}^i(\Gamma, V) \neq 0$, $\forall i \in \mathbb{Z}$, or*
(3) *$\Upsilon_\Gamma(V) > 0$ and $\widehat{H}^{2i}(\Gamma, V) \neq 0$, $\forall i \in \mathbb{Z}$, or*
(4) *$\Upsilon_\Gamma(V) < 0$ and $\widehat{H}^{2i-1}(\Gamma, V) \neq 0$, $\forall i \in \mathbb{Z}$.*

A key point to notice is that the invariant $\Upsilon_\Gamma(V)$ can be computed using the finite subgroups in Γ, an explicit formula is provided in Proposition 2.8.

As a corollary of the above proposition we have the following nonvanishing theorem analogous to the Nakayama–Rim Theorem for finite groups:

Corollary 5.25. *Under the above hypotheses, we have that if $\widehat{H}^i(\Gamma, V) = 0$ for two values of i not congruent* mod 2 *then $\widehat{H}^*(\Gamma, V) \equiv 0$.*

Finally, the techniques used give:

Theorem 4.20. *Let Γ be an $\mathscr{X}$-group, $\Gamma' \lhd \Gamma$ a torsion-free subgroup of finite index (not necessarily of prime order as before), and let V be a finitely generated $\mathbb{Z}$-torsion-free Γ-module. Then either $\widehat{H}^*(\Gamma, V) \equiv 0$ or $\widehat{H}^i(\Gamma, V) \neq 0$ for infinitely many $i \in \mathbb{Z}$.*

The techniques we use are topological in nature, based on the methods introduced by Adem in [2, 3]. However, we deal with the case of *nontrivial coefficients* (unlike [3]) and show how these tools apply to several cohomological questions involving infinite groups. The results we obtain are fundamental in understanding the cohomology of infinite groups, based on the much simpler and better understood case of finite groups. Hopefully these ideas will have further applications in both algebra and topology.

1. Preliminaries

Definition 1.1. We say that a group G has *finite cohomological dimension* (written $cd\, G < \infty$) if there is a finite dimensional space of type $K(G, 1)$.

Examples of the above definition include finitely generated nontrivial free groups, finitely many copies of the integers, the fundamental group of a closed connected surface other than the projective space, and the fundamental group of the complement of a knot in the three-dimensional sphere.

The above examples show that models for classifying spaces might be found with rich geometric structure. Also note that in case a group has finite cd, it follows that the group cannot have nontrivial elements of finite order, i.e. it is necessarily a torsion-free group. Hence we need to extend our family so that torsion is included.

Definition 1.2. Let Γ be a discrete group. We say that Γ has finite virtual cohomological dimension (written $vcd\Gamma < \infty$) if Γ contains a subgroup (which we may assume to be normal) $\Gamma' \subseteq \Gamma$ such that

(a) Γ' has finite cd, and

(b) $|\Gamma/\Gamma'| < \infty$.

As examples of such groups we have the so-called arithmetic groups and mapping class groups.

Definition 1.3. A complete resolution over $\mathbb{Z}\Gamma$ is an acyclic complex F_* of projective $\mathbb{Z}\Gamma$-modules, together with an ordinary projective resolution $\varepsilon: P_* \longrightarrow \mathbb{Z}$, such that F_* and P_* coincide in sufficiently high dimensions.

We may now define a suitable cohomology theory for Γ via these complete resolutions.

Definition 1.4. Let F_* be a complete resolution over $\mathbb{Z}\Gamma$, and V a (left) Γ-module. We define the Farrell cohomology of Γ with coefficients in V as

$$\widehat{H}^*(\Gamma, V) := H^*(Hom_{\mathbb{Z}\Gamma}(F_*, V)),$$

where the RHS denotes the (usual) cohomology of the complex $Hom_{\mathbb{Z}\Gamma}(F_*, V)$.

Note that the above definition makes sense once we have the existence of complete resolutions. Assume this for the time being. We will give an explicit construction of such resolutions later on, suitable for our applications. Next, observe that in case Γ is torsion-free, we have that $\widehat{H}^*(\Gamma, V) \equiv 0$, since in this case we may take $F_* \equiv 0$ as our complete resolution. And in case Γ is a finite group, this cohomology theory coincides with the usual Tate cohomology of the group.

We now turn our attention to a geometric approach. Let G be a group. Let X be a G-CW-complex i.e. a CW-complex in which we have defined an action of G and this is by permuting the cells of X. Then the cellular chain $C(X)$ has a natural action of G, and makes this complex a $\mathbb{Z}G$-module in a natural way. Now observe that this complex consists of copies of $\mathbb{Z}$ in each dimension and the action of G is by permuting these summands. Hence we may describe this complex by the following: in each dimension

$n \geq 0$ we have $C_n(X) \approx \bigoplus_{\sigma \in \Sigma_n} \mathbb{Z}G \otimes_{\mathbb{Z}G_\sigma} \mathbb{Z}_\sigma$, where Σ_n denotes a set of representatives of the G-orbits of cells in dimension n, and G_σ is the isotropy of the cell σ, and $\mathbb{Z}_\sigma$ denotes the orientation module associated with σ.

According to our definition, if G is a group of finite cohomological dimension, then there exists a finite dimensional space (which we may assume to be a CW) of type $K(G,1)$. The above discussion suggests the following.

Definition 1.5. Let $\mathscr{X}$ be the class of discrete groups defined as follows: Γ is in $\mathscr{X}$ if there exists a contractible Γ-complex X with the following properties:

(a) Some normal subgroup Γ' of Γ acts freely on X and X/Γ' is finite,

(b) X^H is contractible for every finite subgroup $H \subset \Gamma$.

We will call such a complex an admissible space for Γ, and we will refer to members of $\mathscr{X}$ as $\mathscr{X}$-groups.

Let $\mathscr{V}$ be the family of finite vcd groups, by a theorem of Serre $\mathscr{X} < \mathscr{V}$ (see [7, Theorem 3.1]). Moreover, by the work of Borel and Serre [5], the class of arithmetic groups is contained in $\mathscr{X}$.

Our next proposition gives us a geometric description of the Farrell cohomology of $\mathscr{X}$-groups Γ in terms of the admissible complex X and the family of finite subgroups of Γ. First we need a few known constructions.

Let G be the quotient Γ/Γ'. As we know, there exists a complete resolution F_* over $\mathbb{Z}G$. We let Γ act on this complex by the canonical map $\Gamma \to G$. Next observe that as X is an $E\Gamma'$ space, the complex $C_*(X)$ provides a resolution over $\mathbb{Z}\Gamma'$. Thus Γ acts diagonally on the complex $F_* \otimes C_*(X)$. It follows from these observations that $F_* \otimes C_*(X)$ is a complete resolution for Γ in the sense previously defined. Note that this proves the existence of a complete resolution for any $\mathscr{X}$-group. Now using this resolution to compute the Farrell cohomology of Γ we have:

Proposition 1.6. *For any $\mathscr{X}$-group Γ, and Γ-module V we have the following isomorphism*:

$$\widehat{H}^*(\Gamma, V) \cong \widehat{H}^*(G, C^*(X) \otimes_{\Gamma'} V).$$

Proof. As indicated above, we use the complete resolution $C_*(X) \otimes F_*$ to compute the Farrell cohomology and the following natural isomorphism:

$$Hom_\Gamma(F_* \otimes C_*(X), V) \cong Hom_G(F_*, C^*(X) \otimes_{\Gamma'} V). \qquad \square$$

2. A numerical invariant

We introduce the following invariant for G-cochain complexes of finite type, i.e. a finite dimensional complex with totally finite rational homology, for a finite group G. Notice that the Euler characteristic χ of a complex of finite homological type is defined.

Definition 2.7. Let C^* be a cochain G-complex of finite homological type and G a finite group. We define the following invariant for C^*:

$$\Upsilon_G(C^*) = |G|\chi((C^* \otimes \mathbb{Q})^G) - \chi(C^* \otimes \mathbb{Q}).$$

Under our assumptions, it is clear that the following examples are of finite type. Let $C^* = C^*(X/\Gamma'; \mathbb{Z})$ where X is an admissible space for the $\mathscr{X}$-group Γ, then

$$\Upsilon_G(C^*) = |\Gamma : \Gamma'|(\tilde{\chi}(\Gamma) - \chi(\Gamma)).$$

This is the difference between the topological Euler characteristic and the group–theoretic version (see [7]).

As a second example, consider

$$C^* = \begin{cases} V & \text{for } * = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where V is a finitely $\mathbb{Z}$–generated and $\mathbb{Z}$–torsion-free G-module. Then

$$\Upsilon_G(C^*) = |G|\, rank_{\mathbb{Z}}\, V^G - rank_{\mathbb{Z}}\, V.$$

We denote this by $\Upsilon_G(V)$.

We will apply this invariant to the cochain complex given by $C^* = C^*(X) \otimes_{\Gamma'} V$, for an $\mathscr{X}$-group Γ and a Γ-module V. The following proposition allows us to compute it explicitly when V is as above.

Proposition 2.8.

$$\Upsilon_G(C^*) = |\Gamma : \Gamma'| \left(\sum_i (-1)^i \left(\sum_{\sigma \in o(i)} \frac{\Upsilon_{\Gamma_\sigma}(V)}{|\Gamma_\sigma|} \right) \right),$$

where $o(i)$ is the number of cells mod Γ *in dimension i, Γ_σ is the isotropy of the cell σ.*

Proof. First apply the decomposition described previously for the complex $C_*(X)$; this gives that in each dimension one has

$$C_i(X) \cong \bigoplus_{\sigma \in o(i)} \mathbb{Z}[\Gamma/\Gamma\sigma],$$

where $o(i)$ is as before, and we are taking one cell in each class mod Γ. Hence for any Γ-module V, and each dimension i we have

$$\begin{aligned} Hom_\Gamma(C_*(X), V) &\cong \bigoplus_{\sigma \in o(i)} Hom_\Gamma(\mathbb{Z}[\Gamma/\Gamma_\sigma], V) \\ &\cong \bigoplus_{\sigma \in o(i)} V^{\Gamma_\sigma}. \end{aligned}$$

Therefore, our invariant can be written as

$$\Upsilon_G(C^*) = |\Gamma : \Gamma'| \sum_{\sigma\in o(i)} (-1)^i dim_{\mathbb{Q}} V^{\Gamma_\sigma} - \sum_{o'(i)} (-1)^i dim_{\mathbb{Q}} V,$$

where this time $o'(i)$ is the number of cells mod Γ'. Now, by using the fundamental counting principle for groups acting on sets we have that the above equals

$$|\Gamma : \Gamma'| \sum_{\sigma\in o(i)} (-1)^i dim_{\mathbb{Q}} V^{\Gamma_\sigma} - \sum_{\sigma\in o(i)} (-1)^i dim_{\mathbb{Q}} V \cdot |G : G_\sigma|$$

$$= |\Gamma : \Gamma'| \left(\sum_{\sigma\in o(i)} (-1)^i \left(dim_{\mathbb{Q}} V^{\Gamma_\sigma} - \frac{dim_{\mathbb{Q}} V}{|\Gamma_\sigma|} \right) \right);$$

this last term is our desired result. □

Definition 2.9. Let Γ be a $\mathscr{X}$-group, and V a Γ-module as above. We define

$$\Upsilon_\Gamma(V) := \Upsilon_G(C^*(X) \otimes_{\Gamma'} V),$$

where the RHS is as in the previous proposition. Note that it is determined on the finite subgroups of Γ.

To finish this section we mention that an immediate consequence of the definition of our invariant is that given an exact sequence of G-cochain complexes of finite type, $0 \to A^* \to C^* \to B^* \to 0$ we have that $\Upsilon_G(C^*) = \Upsilon_G(A^*) + \Upsilon_G(B^*)$.

3. Minimal resolutions

In this section we develop algebraic tools in order to establish the main theorem of this paper. We start by recalling some basic definitions. Throughout this section we will assume that G is a finite p-group for some prime p. Given a G-module M a projective resolution of M over $\mathbb{Z}G$ is an exact sequence of the form $\cdots P_i \to P_{i-1} \to \cdots \to P_0 \to M \to 0$, where all of the P_i's are $\mathbb{Z}G$-projective G-modules. We will assume that M is a finitely $\mathbb{Z}$–generated $\mathbb{Z}$–torsion-free G-module. We will also say that a projective resolution of M is *minimal* if P_n is a projective resolution of minimal rank mapping onto $ker\, \partial_{n-1}$, for all $n \geq 0$. We also recall that given any G-module M, and for any integer n, there exists a $\mathbb{Z}$-torsion-free module $\Omega(M)$ such that

$$\widehat{H}^*(G, \Omega^i(M)) \cong \widehat{H}^{*-i}(G, M).$$

The above module Ω is known as a dimension shifting of M. Our first result gives conditions to a resolution in order to be minimal. The following appeared in [2]:

Proposition 3.10. *Let M be a finitely $\mathbb{Z}$–generated $\mathbb{Z}$-torsion-free G-module. Let $C = (P_*, \partial_*)$ be a projective resolution of M. Then C is minimal if and only if $rank_{\mathbb{Z}} P_n = |G|\, dim_{\mathbb{F}_p} H^n(G, (M \otimes \mathbb{F}_p)^*)$, where * denotes the usual dual.*

At this point, observe that we have already found the cohomology of G with coefficients in a cochain complex. In order to use a result such as the previous one we either have to develop a theory of minimal resolutions for cochain complexes or find a module that encloses the same cohomological information as the given cochain. The following lemma gives us such a module.

Proposition 3.11. *Let* $\{C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} C^n\} = (C^*, \delta)$ *be a cochain complex of finitely generated* $\mathbb{Z}G$*-modules. Then* C^* *fits into an exact sequence*

$$0 \to C^* \to D^* \to F^* \to 0$$

of f.g. $\mathbb{Z}G$*-cochain complexes where each* F^i *is* $\mathbb{Z}G$*-free, and* $H^*(D) = 0$ *for* $* > 0$.

Proof. To begin, we choose a f.g. free $\mathbb{Z}G$-module F^{n-1} which maps onto C^n under a map $f^{n-1}: F^{n-1} \longrightarrow C^n$. Let $\Delta^{n-1}: C^{n-1} \oplus F^{n-1} \longrightarrow C^n$ be defined by $\Delta^{n-1}(x, y) = \delta^{n-1}(x) + f^{n-1}(y)$. The composition

$$C^{n-2} \hookrightarrow C^{n-2} \oplus 0 \xrightarrow{(\delta^{n-2},0)} C^{n-1} \oplus F^{n-1} \xrightarrow{\Delta^{n-1}} C^n$$

is zero. Now choose F^{n-2}, $f^{n-2}: F^{n-2} \longrightarrow \ker \Delta^{n-1}$ onto, and define $\Delta^{n-2}: C^{n-2} \oplus F^{n-2} \to C^{n-1} \oplus F^{n-1}$ by $\Delta^{n-2}(x,y) = (\delta^{n-2}(x), 0) + f^{n-2}(y)$. Note that

$$\Delta^{n-1} \circ \Delta^{n-2}(x,y) = \Delta^{n-1}((\delta^{n-2}(x), 0) + F^{n-2}(y) = 0.$$

Note that the diagram

$$\begin{array}{ccccc} C^{n-2} \oplus F^{n-2} & \xrightarrow{\Delta^{n-2}} & C^{n-1} \oplus F^{n-1} & \xrightarrow{\Delta^{n-1}} & C^n \quad \to 0 \\ {\scriptstyle (1,0)}\uparrow & & {\scriptstyle (1,0)}\uparrow & & {\scriptstyle 1}\uparrow \\ C^{n-2} & \xrightarrow{\delta^{n-2}} & C^{n-1} & \xrightarrow{\delta^{n-1}} & C^n \end{array}$$

clearly commutes. Also by construction it is clear that the top row is exact. Continuing like this we obtain a commutative diagram with exact top row

$$\begin{array}{cccccccccc} 0 & \to & H^0 & \to & C^0 \oplus F^0 & \xrightarrow{\Delta^0} & \cdots & \to & C^{n-1} \oplus F^{n-1} & \xrightarrow{\Delta^{n-1}} & C^n \quad \to 0 \\ & & & & \uparrow & & \cdots & & \uparrow & & \uparrow \\ & & & & C^0 & \to & \cdots & \to & C^{n-1} & \to & C^n \end{array}$$

Let $D^* = (C^* \oplus F^*, \Delta^*)$; then $C^* \hookrightarrow D^*$, and clearly D^*/C^* is a cochain complex of f.g. free $\mathbb{Z}G$-modules. By construction, the cohomology of D^* is concentrated in dimension 0. □

The main idea of the above proposition is that it allows us to get cohomological information of the group G with coefficients in a cochain complex from the cohomology of the group with coefficients in a $\mathbb{Z}G$-lattice: from the long exact sequence associated with $0 \to C^* \xrightarrow{i} D^* \to F^* \to 0$ and the fact that $\widehat{H}^*(G, F^*) \equiv 0$ as each F^i is $\mathbb{Z}G$-free, we obtain that $i^*: \widehat{H}^*(G, C^*) \to \widehat{H}^*(G, D^*)$ is an isomorphism. On the other hand, all the cohomology of D^* is concentrated in dimension zero, hence the spectral sequence

$$\widehat{H}^p(G, H^q(D^*)) \Longrightarrow \widehat{H}^{p+q}(G, D^*)$$

degenerates to yield the isomorphism

$$\widehat{H}^i(G, D^*) \cong \widehat{H}^i(G, H^o(D^*)) \quad \forall i \in \mathbb{Z},$$

and the following definition.

Definition 3.12.

$$M_{C^*} := H^0(D^*),$$

where D^* is as in Proposition 3.11.

We will now apply the above construction and the notion of minimal resolutions when we have coefficients in a cochain complex. The idea is now clear: we want to introduce the notion of minimal resolutions of cochain complexes. Since we are only dealing with the cohomological properties of these complexes, it is sufficient to get a $\mathbb{Z}G$-lattice that encodes the same cohomological information. In other words, for any cochain G-complex C^*, let M_{C^*} be as in the last definition. Let $P_* \to M_{C^*}$ be a resolution for M_{C^*}. Then using Theorem 3.10 and the above discussion we have

Proposition 3.13. *Under the above hypotheses, the resolution $P_* \to M_{C^*}$ is minimal if and only if*

$$rank\, P_* = \begin{cases} |G|\, dim(M_{C^*} \otimes \mathbb{F}_p)^G & \text{for } i = 0, \\ |G|\, dim\, \widehat{H}^{-i-1}(G, C^* \otimes \mathbb{F}_p) & \text{for } i > 0. \end{cases}$$

4. Exponents and cohomology

In this section we will use exponents to prove that cohomological nonvanishing in a large enough range implies it in all dimensions. To begin we recall:

Definition 4.14. Let A be a finite abelian group. The exponent $exp(A)$ as the minimum natural number n such that $n \cdot A = 0$.

We know that in case G is a finite group acting on a finite dimensional space X, we have that $|G| \cdot \widehat{H}^*_G(X) \equiv 0$. This provides us with interesting examples of torsion groups. We analyze the exponent in this setup. We prove a fundamental result based on a theorem of Browder [6]:

Theorem 4.15. *Let X be a free, connected finite G-CW-complex, and M a finitely $\mathbb{Z}$–generated $\mathbb{Z}$–torsion-free $\mathbb{Z}G$–module. Then*

$$exp\,\widehat{H}^0(G,M) \mid \prod_{i\geq 1} exp(\widehat{H}^{-i-1}(G;H^i(X;\mathbb{Z})\otimes M)).$$

Proof. There is a spectral sequence with coefficients of the form

$$E_2^{p,q} = \widehat{H}^p(G;H^q(X;\mathbb{Z})\otimes M) \Rightarrow \widehat{H}_G^{p+q}(X;M) \equiv 0.$$

As $E_2^{*,q} = 0$ for $q < 0$ we have exact sequences

$$E_{r+1}^{-r-1,r} \to E_{r+1}^{0,0} \to E_{r+2}^{0,0} \to 0,$$

for $r = 1,\dots,dim(X)$. Hence by the property of the exponent described above, we have

$$\frac{exp(E_{r+1}^{0,0})}{exp(E_{r+1}^{-r-1,r})} \mid exp(E_{r+2}^{0,0}).$$

Next by multiplying all of these out we obtain

$$exp(\widehat{H}^0(G;M)) \mid \prod_{r\geq 1} exp(E_{r+1}^{-r-1,r}).$$

Now recall that the $E_{r+1}^{-r-1,r}$ terms are subquotients of $E_2^{-r-1,r} = \widehat{H}^{-r-1}(G;H^r(X)\otimes M)$ from which we infer the result. □

We now apply this to a suitable G- space X, namely: let $G \hookrightarrow U(n)$ be an embedding of G into a unitary group for some n. This provides a free action of G (by left translations) on $X = U(n)$. Moreover, this action is homologically trivial as it extends to an action of all of $U(n)$ and this is a path–connected compact Lie group. Using this situation we have the following

Corollary 4.16. *Let $G \hookrightarrow U(n)$ be an embedding of G into a unitary group $U(n)$, for some n. Let M be a f.g. torsion-free $\mathbb{Z}G$ module then*

$$exp(\widehat{H}^k(G;M)) \mid \prod_{r=1}^{n^2} exp(\widehat{H}^{r+k+1}(G;M)).$$

Proof. Recall that $H^*(U(n)) \cong H^*(\mathbb{S}^1 \times \mathbb{S}^3 \times \cdots \times \mathbb{S}^{2n-1})$, and the action of G is homologically trivial. Hence,

$$\prod_{r=1}^{\infty} exp\,\widehat{H}^{-r-1}(G,H^r(X)\otimes M) \cong \prod_{r=1}^{n^2} exp\,\widehat{H}^{-r-1}(G,M).$$

Now, by Tate duality $\widehat{H}^r(G) \cong \widehat{H}^{-r}(G,M')$, where M' denotes the dual of M, we have that the last product is that of $\widehat{H}^{r+1}(G,M)$ for $r = 1,\dots,n^2+1$. Consequently,

$$exp\,\widehat{H}^0(G,M) \mid \prod_{r=1}^{n^2} exp\,\widehat{H}^{r+1}(G,M).$$

Now for an arbitrary index k, let $\Omega^{-k}M$ be a -k-dimensional dimension shift of M: i.e. we have $\widehat{H}^0(G,\Omega^{-k}M) \cong \widehat{H}^k(G,M)$. Apply the previous result to the module $\Omega^{-k}M$ and observe that $\widehat{H}^{r+1}(G,\Omega^{-k}M) \cong \widehat{H}^{r+1+k}(G,M)$ to get the statement. □

The following corollary has been proved by purely algebraic methods in [4]. We emphasize that our proof relies on the degree of a faithful representation of the finite group G and the divisibility properties of the exponent.

Corollary 4.17. *Under the hypotheses of the above corollary, assume that, for n^2+2 consecutive values of r, the groups $\widehat{H}^r(G,M)$ vanish. Then $\widehat{H}^*(G,M) \equiv 0$.*

Proof. Assume that $\widehat{H}^{r+k+1}(G,M) = 0$ for $r = 0,1,\cdots,n^2$; by the previous corollary we have that $\widehat{H}^k(G,M) = 0$. Thus $\widehat{H}^r(G,M) = 0$ for all $r \leq n^2+k+1$. Now use Tate duality $\widehat{H}^{-k}(G,M) = \widehat{H}^k(G,M')$ and the same argument again to get that $\widehat{H}^r(G,M) = 0$ for all $r \geq n^2+k+2$. □

As an application of these results we have the following:

Theorem 4.18. *Let $\varphi : M \to N$ be a map of $\mathbb{Z}G$-modules such that the induced map in cohomology*

$$\varphi_* : \widehat{H}^k(G,M) \to \widehat{H}^k(G,N)$$

is an isomorphism for n^2+2 consecutive values of k. Then the induced map φ_* is an isomorphism for all values of k.

Proof. Let P be a finitely generated projective module that maps onto M. Let $\tau : M \oplus P \to N$ defined as the sum of later homomorphism and φ. Notice that $\widehat{H}^*(G,M) \cong \widehat{H}^*(G,M \oplus P)$. Then τ is an onto map , $\tau_* = \varphi_*$, hence τ_* is an isomorphism for n^2+1 consecutive values of k. Let K be the kernel of τ, the long exact sequence in Tate cohomology associated with the short exact sequence $0 \to K \to M \to N \to 0$ gives that $\widehat{H}^k(G,K) = 0$ for n^2+1 consecutive values of k. Thus, by corollary (4.17), $\widehat{H}^*(G,K) \equiv 0$. Thus φ_* is an isomorphism for all k. □

Corollary 4.19. *Let $\varphi : M \to N$ be a map of $\mathbb{Z}\Gamma$-modules such that the induced map in cohomology*

$$\varphi_* : \widehat{H}^k(\Gamma,M) \to \widehat{H}^k(\Gamma,N)$$

is an isomorphism for n^2+2 *consecutive values of* k. *Then the induced map* φ_* *is an isomorphism for all values of* k.

Theorem 4.20. *Let* Γ *be an* $\mathscr{X}$*-group, let* $\Gamma' \lhd \Gamma$ *be a torsion free subgroup of finite index (not necessarily of prime order as before), and let* V *be a finitely* $\mathbb{Z}$*–generated* $\mathbb{Z}$*-torsion-free* Γ*-module. Then either* $\widehat{H}^*(\Gamma, V) \equiv 0$ *or* $\widehat{H}^i(\Gamma, V) \neq 0$ *for infinitely many* $i \in \mathbb{Z}$.

Proof. By the work done in previous section we know that $\widehat{H}^*(\Gamma, V)$ is isomorphic to the Tate cohomology of the finite group $G := \Gamma/\Gamma'$ with coefficients in a suitable cochain complex. We now apply the proposition (4.17) to this case. □

5. Cohomological nonvanishing

In this section we establish our nonvanishing cohomological result. G will be a finite p–group. We start with some remarks: Let C^* be a G-cochain complex of finite type. Let $MC := M_{C^*}$ be as in Definition 3.12. Now consider a finite stage of a minimal resolution $P_* \to MC$:

$$0 \to \Omega^{k+1}(MC) \to P_k \to \cdots \to P_0 \to MC \to 0.$$

Then according to (3.13) we have that

$$\begin{aligned}&(-1)^k\, rank\, \Omega^{k+1}(MC) + rank(MC)\\ &\quad = |G|\left[\sum_{i=1}^{k}(-1)^i dim_{\mathbb{F}_p} \widehat{H}^{-i-1}(G, C^* \otimes \mathbb{F}_p) + dim_{\mathbb{F}_p}(MC' \otimes \mathbb{F}_p)^G\right].\end{aligned}$$

On the other hand, from the long exact sequence induced in cohomology by the short exact sequence of G-cochain complexes,

$$0 \to C^* \xrightarrow{\cdot p} C^* \to C^*_p \to 0,$$

we obtain

$$dim_{\mathbb{F}_p} \widehat{H}^{-i-1}(G, C^* \otimes \mathbb{F}_p) = dim_{\mathbb{F}_p} \widehat{H}^{-i-1}(G, C^*) \otimes \mathbb{F}_p + dim_{\mathbb{F}_p} \widehat{H}^{-i}(G, C^*) \otimes \mathbb{F}_p,$$

and that

$$dim_{\mathbb{F}_p}(MC \otimes \mathbb{F}_p)^G = dim_{\mathbb{F}} \widehat{H}^{-1}(G, C^*) \otimes \mathbb{F}_p + rank(MC^G).$$

Combining the above equalities we finally obtain

$$\begin{aligned}(-1)^k rank\, \Omega^{k+1}(MC) &= (-1)^k |G| dim\, \widehat{H}^{-k}(G, C^*) \otimes \mathbb{F}_p + \Upsilon_G(MC)\\ &= (-1)^k |G| dim\, \widehat{H}^{-k}(G, C^*) \otimes \mathbb{F}_p + \Upsilon_G(C^*),\end{aligned}$$

where the last equality follows from the property of the Υ-invariant described in [3], and the definition of Ω. Summarizing we have the following

Proposition 5.21.

$$(-1)^k rank_{\mathbb{Z}}\, \Omega^{k+1}(MC) = (-1)^k |G| dim_{\mathbb{F}_p} \widehat{H}^{-k}(G, C^*) \otimes \mathbb{F}_p + \Upsilon_G(C^*).$$

Let us now assume that for a given G-complex C^*, the module MC is not $\mathbb{Z}G$-projective, hence $\Omega^j(MC) \neq 0\ \forall j$. This gives the following

Corollary 5.22. *Let G be a finite p-group, and C^* a G-complex of finite type as above. Then*

$$|G| \cdot dim\, \widehat{H}^i(G, C^*) \otimes \mathbb{F}_p + (-1)^{j-1} \Upsilon_G(C^*) > 0 \quad \forall j \in \mathbb{Z}.$$

Now apply Proposition 5.22 in case $C^* = C^*(X) \otimes_{\Gamma'} V$ where X is an admissible complex for the $\mathscr{X}$-group Γ and V is a Γ-module. We have the following

Theorem 5.23. *Let Γ be an $\mathscr{X}$-group with $\Gamma' \lhd \Gamma$ a torsion free normal subgroup of finite index. Let $G := \Gamma/\Gamma'$. Let V be a finitely $\mathbb{Z}$-generated, and $\mathbb{Z}$-torsion free Γ-module. Assume that Γ is not torsion free and that $|G| = p^n$ for some prime p. Then either*

$$dim_{\mathbb{F}_p} \widehat{H}^i(\Gamma, V) \otimes \mathbb{F}_p + (-1)^{i+1} \frac{\Upsilon_\Gamma(V)}{|\Gamma : \Gamma'|} > 0 \quad \forall i \in \mathbb{Z},$$

or

$$\widehat{H}^*(\Gamma, V) = 0 \quad \forall * \in \mathbb{Z}.$$

As an application of this result we have the following which describes a global nonvanishing of these Farrell groups:

Theorem 5.24. *Under the hypotheses of the above theorem, precisely one of the following must hold:*

(1) *$\Upsilon_\Gamma(V) = 0$ and V is Γ-cohomologically trivial, or*
(2) *$\Upsilon_\Gamma(V) = 0$ and $\widehat{H}^i(\Gamma, V) \neq 0,\ \forall i \in \mathbb{Z}$, or*
(3) *$\Upsilon_\Gamma(V) > 0$ and $\widehat{H}^{2i}(\Gamma, V) \neq 0,\ \forall i \in \mathbb{Z}$, or*
(4) *$\Upsilon_\Gamma(V) < 0$ and $\widehat{H}^{2i-1}(\Gamma, V) \neq 0,\ \forall i \in \mathbb{Z}$.*

Proof. The proof is immediate from the above proposition. □

Notice that in case $\Upsilon_H(V) = 0$ for every finite subgroup H of Γ, then by Proposition 2.8 we would have $\Upsilon_\Gamma(V) = 0$. In fact it is enough to analyze this invariant on conjugacy classes of the finite subgroups of Γ.

As a corollary of the above proposition we have the following result which is an extension to discrete groups of classical work due to Nakayama and Rim [10, 12].

Corollary 5.25. *Under the above hypotheses, we have that if* $\widehat{H}^i(\Gamma, V) = 0$ *for two values of i not congruent* mod 2 *then* $\widehat{H}^*(\Gamma, V) \equiv 0$.

We finish this section with another application of the techniques used in this chapter to prove a special case of a result recently proved by [9]. Lee's result is an extension of a similar statement concerning the cohomology of finite groups, to the family of $\mathscr{X}$-groups. We mention that Lee's methods heavily rely on homotopy theory and applications of the Segal conjecture which algebraically identifies the localized classifying space of a finite group.

Throughout the rest of the section, p will denote a fixed prime. Let $\mathscr{P}(\Gamma)$ be the category whose objects are the finite p-subgroups of Γ, and whose morphisms are given by conjugation and inclusions. We also recall that for any $\Gamma-\mathrm{CW}$ space Z we have a fibration:

$$\begin{array}{ccc} Z & \rightarrow & E\Gamma \times_\Gamma Z \\ & & \downarrow \\ & & B\Gamma \end{array}$$

Finally, recall that in case that the action has fixed points, i.e. $Z^\Gamma \neq \emptyset$, there is a section $\widehat{H}^*(\Gamma) \hookrightarrow \widehat{H}^*_\Gamma(Z)$.

Our following lemma is an application of Mackey's formula [14] for the description of the process of induction-restriction operations on G-modules:

Lemma 5.26. *Let Y be a finite dimensional H-space where H is a subgroup of G, and G is a finite group. Let* $Y \times_H G$ *be the induced action to G, and* $P \subset G$ *a p subgroup. Then we have an isomorphism*

$$\widehat{H}^*_P(Y \times_H G) \longrightarrow \bigoplus_{s \in E} \widehat{H}^*_{H^s \cap P}(Y),$$

where E is a coset representative for the double classes $H \backslash G / P$.

Theorem 5.27. *Let* $\varphi : \Gamma_2 \to \Gamma_1$ *be a homomorphism of* $\mathscr{X}$*-groups such that*

$$\varphi^* : \widehat{H}^*(\Gamma_1) \to \widehat{H}^*(\Gamma_2)$$

is an isomorphism with trivial $\mathbb{Z}_p$*-coefficients. Assume further that both groups have normal, torsion–free subgroups of index a finite power of a prime p. Then* φ *induces a bijection between the objects in* $\mathscr{P}(\Gamma_2)$ *and* $\mathscr{P}(\Gamma_1)$.

Proof. We first prove that φ induces an injection of categories: Let X be an admissible complex for Γ_1 and make Γ_2 act on X via φ. Let $K = ker(\varphi)$. Our goal is to show that K has no subgroup P of order p^n. Let $G \hookrightarrow U$ be an embedding of G into a unitary group $U = U(n)$ for some n and let G act on U via this map. Therefore, Γ_1 acts freely on the finite dimensional space $X \times U$, and Γ_2 also acts on this space, not necessarily freely. Since the map φ^* is an isomorphism, it induces an isomorphism between the E_2 terms of Leray the spectral sequences given by the corresponding fibrations [8]. Thus, we have an isomorphism of the corresponding abutments and this gives an isomorphism:

$$0 \equiv \widehat{H}^*_{\Gamma_1}(X \times U) \cong \widehat{H}^*_{\Gamma_2}(X \times U).$$

This means any p–elementary abelian subgroup of Γ_2 must act freely on $X \times U$, hence K must be p–torsion-free [7].

From the above we may assume that φ is an injection. Notice that in case Γ_1 is torsion-free there is nothing to prove. Thus, we may assume that Γ_1 is not torsion free. Furthermore, as we have assumed that φ is the inclusion, we may take the same admissible complex X for both Γ_1 and Γ_2. Thus we have the extensions $1 \to \Gamma' \to \Gamma_1 \to G \to 1$ and $1 \to \Gamma' \cap \Gamma_2 \to \Gamma_2 \to H \to 1$. Now let $P \subset G$ be a subgroup which is the image of a finite subgroup in Γ_1. According to the isomorphism given in Proposition 1.6 we have the following commutative diagram:

$$\begin{array}{ccc} \widehat{H}^0_G(X/\Gamma') & \xrightarrow{\mu} & \widehat{H}^0_G(Y \times_H G) \\ {\scriptstyle res_P}\downarrow & & \downarrow{\scriptstyle res_P} \\ \widehat{H}^0_P(X/\Gamma') & \xrightarrow{\mu'} & \bigoplus_{s \in E} \widehat{H}^0_{H^s \cap P}(Y) \end{array}$$

where $Y = X/\Gamma' \cap \Gamma_2$, H^s denotes conjugation by s, and μ is the composition $ind \cdot \varphi^*$, where $ind : \widehat{H}^*_H(Y) \to \widehat{H}^*_G(Y \times_H G)$ is the induction isomorphism. The morphism μ' is induced by μ and is also an isomorphism as μ is and G is a p-group [10]. Both the left and right morphisms are induced by the corresponding restrictions to the p-group $P \subseteq G$. We also have that $(X/\Gamma')^P$ is not empty. Hence, we have an injection $\mathbb{Z}/|P| \hookrightarrow \widehat{H}^0_P(X/\Gamma')$, thus $\mathbb{Z}/|P| \hookrightarrow \bigoplus_{s \in E} \widehat{H}^0_{H^s \cap P}(Y)$, and this implies that there exists $s \in G$ such that $|P| = |H^s \cap P|$, i.e. there is an s' such that $P^{s'} \subseteq H$, hence φ is onto on $\mathscr{P}$. $\square$

The above argument works as we are able to prove that μ' is an isomorphism if μ is. The above argument does not extend, at least immediately, to arbitrary finite groups. It would be desirable to be able to prove the above statement, for general finite groups, using only the techniques presented throughout this section.

6. Examples

We conclude with some applications of the results obtained. Assume that Γ is a group that fits into an extension $1 \to \Gamma' \to \Gamma \to P \to 1$, where Γ' is an f.g. free group, and P is a p-group for some prime p. Let V be the trivial module $\mathbb{Z}$. Let $\tilde{c} = dim\, H^1(\Gamma, \mathbb{Q})$, $c = dim\, H^1(\Gamma', \mathbb{Q})$. Then since $H^1(\Gamma, \mathbb{Q}) = H^1(\Gamma', \mathbb{Q})^G$, and Γ' is a free group, we may take an admissible complex of dimension one, i.e. a graph. Hence the value of $\Upsilon := \Upsilon_\Gamma(\mathbb{Z})$ depends only on the values of c and $\tilde{c}$, and is

$|P|(1-\tilde{c})-(1-c)$. Thus its sign is determined as follows:

(a) If $c=1=\tilde{c}$ then $\Upsilon=0$ hence $\widehat{H}^{2i}\neq 0$ for every $i\in\mathbb{Z}$.

(b) If $c=1$ and $\tilde{c}=0$ then $\Upsilon=|P|$. Hence $\widehat{H}^{2i}>|P|$ for every $i\in\mathbb{Z}$.

(c) If $c>1$ and $\tilde{c}=1$ then $\Upsilon>0$ hence $\widehat{H}^{2i}\neq 0$ for every $i\in\mathbb{Z}$.

(d) If $c>1$ and $\tilde{c}>1$ the sign of Υ is determined by

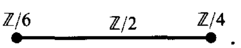
$$\text{if}\begin{cases}|P|\leq(1-c)/(1-\tilde{c}) & \text{Then } \Upsilon>0 \text{ and } \widehat{H}^{2i}\neq 0,\forall i\in\mathbb{Z}.\\ |P|>(1-c)/(1-\tilde{c}) & \text{Then } \Upsilon<0 \text{ and } \widehat{H}^{2i+1}\neq 0,\forall i\in\mathbb{Z}.\end{cases}$$

As an application of the above, let $P=\mathbb{Z}/2(\alpha)\oplus\mathbb{Z}/2(\beta)$ and let X be a wedge of four circles, $e_1,\ldots,e_4$ with common point x. Let P act cellwise on X in the only possible way on the point x and on the 1 cells as $\alpha\colon e_1\leftrightarrow e_2$, and fixing e_3 and e_4, and $\beta\colon e_3\leftrightarrow e_4$, and fixing e_1 and e_2. Let $X\times_P EP$ be the Borel construction on the space X, then from the bundle given by

$$\begin{array}{ccc} X & \to & X\times_P EP\\ & & \downarrow\\ & & BP\end{array}$$

we have that $\Gamma:=\pi_1(X\times_P EP)$ fits into an extension $1\to\Gamma^4\to\Gamma\to P\to 1$, where Γ^4 is the fundamental group of X and hence it is free in four generators. Therefore, and following the previous notation, $\tilde{c}=2$, $c=4$ and hence $\Upsilon=-1$. Thus, $\widehat{H}^{2i+1}(\Gamma,\mathbb{Z})\neq 0$ for every $i\in\mathbb{Z}$.

As a second example consider $\Gamma\subset SL_2(\mathbb{Z})$ determined by the extension $1\to\Gamma(3)\to\Gamma\to Q_8\to 1$ where $\Gamma(3)$ is the level 3 congruence subgroup, and Q_8 is the 2-Sylow subgroup of $SL_2(\mathbb{F}_3)$. We already know that $\chi(SL_2(\mathbb{Z}))=-\frac{1}{12}$ and that $|SL_2(\mathbb{F}_3)|=24$, thus $\Gamma(3)$ is free of rank 3. We also know from a work of Serre [15] that $SL_2(\mathbb{Z})\cong\mathbb{Z}/6*_{\mathbb{Z}/2}\mathbb{Z}/4$ and it acts on a tree with orbit space



Moreover, we have that $\Gamma(3)$ acts freely on this tree with orbit space a wedge of three circles. Then this space has an action of $SL_2(\mathbb{F}_3)$ with isotropy as above; thus its cellular complex has the form

$$\mathbb{Q}[SL_2(\mathbb{F}_3)/\mathbb{Z}/2]\to\mathbb{Q}[SL_2(\mathbb{F}_3)/\mathbb{Z}/4]\oplus\mathbb{Q}[SL_2(\mathbb{F}_3)/\mathbb{Z}/6]\to\mathbb{Q}\to 0.$$

Next by restricting to $Q_8\subset SL_2(\mathbb{F}_3)$ and using Mackey's formula we get

$$(\mathbb{Q}[Q_8/\mathbb{Z}/2])^3\to(\mathbb{Q}[Q_8/\mathbb{Z}/4])^3\oplus\mathbb{Q}[Q_8/\mathbb{Z}/2]\to\mathbb{Q}\to 0.$$

Let V be a $\mathbb{Z}$-torsion-free Q_8-module. We let Γ act on V through the projection map $\Gamma\to Q_8$. Thus, according to the formula given in Definition 2.8 we have that

$$\begin{aligned}\tfrac{1}{8}\Upsilon_\Gamma(V)&=-3\left(\tfrac{1}{2}\Upsilon_{\mathbb{Z}/2}(V)\right)+3\left(\tfrac{1}{4}\Upsilon_{\mathbb{Z}/4}(V)\right)+\tfrac{1}{2}\Upsilon_{\mathbb{Z}/2}(V)\\ &=-\Upsilon_{\mathbb{Z}/2}+\tfrac{3}{4}\Upsilon_{\mathbb{Z}/4}(V)\\ &=-(2\,rank\,V^{\mathbb{Z}/2}-rank\,V)+3\left(rank\,V^{\mathbb{Z}/4}-\tfrac{1}{4}\,rank\,V\right)\end{aligned}$$

Hence, in case $V = \mathbb{Z}$ as a trivial Γ-module we have that

$$\tfrac{1}{8}\Upsilon_\Gamma = \tfrac{5}{4}$$

and it follows that

$$dim\, \widehat{H}^{2i}(\Gamma, \mathbb{Z}) \geq 2 \quad \text{for every } i \in \mathbb{Z}.$$

On the other hand, let $V = \mathbb{Q}^2$, and let Γ act via the projection of Q_8 onto $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Let X, Y be two generators of order four for Q_8. Hence we have a projection $Q_8 \to \overline{X}\mathbb{Z}/2 \oplus \overline{Y}\mathbb{Z}/2$. Let $\overline{X}$ act trivially on V and $\overline{Y}$ by inversion. From this we have that the action of $\overline{XY}$ is trivial. Keeping track of this information in our formula, it now follows that

$$\tfrac{1}{8}\Upsilon_\Gamma(V) = \tfrac{5}{2},$$

and that

$$dim\, \widehat{H}^{2i}(\Gamma, V) \geq 3 \quad \text{for every } i \in \mathbb{Z}.$$

Acknowledgements

I thank Prof. Alejandro Adem for his advice and constant support. I also thank Prof. M. Isaacs for helpful conversations and the referee for simplifying earlier versions of this paper.

References

[1] A. Adem, Cohomological restrictions on finite group actions, J. Pure Appl. Algebra 54 (1988) 117–139.
[2] A. Adem, Cohomological non-vanishing for modules over p-groups, J. Algebra 141 (2) (1991) 376–381.
[3] A. Adem, Euler characteristics and cohomology of p-local discrete groups, J. Algebra 149 (1) (1992) 183–196.
[4] D. Benson, J. Carlson and G Robinson, On the vanishing of group cohomology, J. Algebra 131 (1990) 40–73.
[5] A. Borel and J.P. Serre, Corners and arithmetic groups, Comment. Math. Helv. 48 (1974) 244–297.
[6] W. Browder, Cohomology and group actions, Invent. Math. 71 (1983) 599–608.
[7] K. Brown, Cohomology of Groups, Graduate Texts in Mathematics, Vol. 87 (Springer, Berlin, 1982).
[8] H. Cartan and S. Eilenberg, Homological Algebra (Princeton University Press, Princeton, NJ, 1956).
[9] C-N. Lee, The stable homotopy type of the classifying space of virtually torsion-free groups, Topology 33 (1994) 721–728.
[10] T. Nakayama, On modules over a finite group I, Illinois J. Math. 1 (1957) 36–43.
[11] D. Quillen, The spectrum of an equivariant cohomology ring I & II, Ann. Math. 94 (1971) 549–602.
[12] D.S. Rim, Modules over finite groups, Ann. Math. 69 (1959) 700–712.
[13] J. Rotman, An Introduction to Homological Algebra (Academic Press, New York, 1979).
[14] J.P. Serre, Linear Representations of Finite Groups, Graduate Texts in Mathematics, Vol. 42 (Springer, Berlin, 1979).
[15] J.P. Serre, Trees (Springer, Berlin, 1980).
[16] R.G. Swan, Minimal resolutions for finite groups, Topology 4 (1965) 193–208.